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Quaternionic Exact Solution to the Relative Orbital Motion Problem

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The nonlinear initial value problem that models the relative orbital Keplerian motion is expressed using the quaternions algebra. Then, this problem is solved and closed-form expressions for the relative position and relative velocity are obtained. The procedure that allows solving completely the relative orbital motion is purely analytic, without any geometrical considerations. The closed-form expressions hold for any chief and deputy inertial trajectories and have no singularities. The solution offered in this paper is presented in a coordinate-free form that allows a variety of expressions, depending on the coordinate system that is chosen and on the orbit elements that one wants to use as independent variables (time, eccentric anomaly, true anomaly).

Nomenclature

$(\mathbf{a}, \mathbf{b}, \mathbf{c})$	=	triple vector product of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$
e	=	eccentricity (dimensionless)
E	=	eccentric anomaly, rad
f	=	true anomaly, rad
$\mathbf{h}$	=	specific angular momentum, m^2/s
n	=	mean motion, rad/s
p	=	semilatus rectum, m
$\mathbf{q}$	=	unit quaternion
$\mathbf{r}$	=	radius vector, m
t	=	time, s
$\mathbf{u}$	=	unit vector
U	=	universal functions
$\mathbf{v}$	=	velocity vector, m/s
w	=	magnitude of vector $\mathbf{w}$
$\boldsymbol{\omega}$	=	angular velocity, rad/s
$\tilde{\boldsymbol{\omega}}$	=	skew-symmetric tensor (matrix) associated to vector $\boldsymbol{\omega}$
μ	=	Earth gravitational parameter, m^3/s^2

I. Introduction

THE relative orbital motion problem may now be considered classic, because of so many scientific papers written on this subject in the last few decades. This problem is also quite important, due to its numerous applications: spacecraft formation flying, rendezvous operations, distributed spacecraft missions.

The analysis of relative motion began in the early 1960s with the paper of Clohessy and Wiltshire [1], who obtained the equations that model the relative motion in the situation in which the chief spacecraft has a circular orbit and the attraction force is not affected by the Earth oblateness. They linearized the nonlinear initial value problem that models the relative motion by assuming that the relative distance between the two spacecraft remains small during the mission. The Clohessy–Wiltshire equations are still used today in rendezvous maneuvers, but they cannot offer a long-term accuracy because of the secular terms present in the expression of the relative position vector. Independently, Lawden [2], Tschauner and Hempel [3], and Tschauner [4] obtained the solution to the linearized equations of motion in the situation in which the chief orbit is elliptic, but their solutions still involved secular terms and also had singularities. The singularities in the Tschauner–Hempel equations were removed firstly by Carter [5] and also by Yamanaka and Andersen [6]. Later on, the formation flying concept began to be considered, and the problem of deriving equations for the relative motion with a long-term accuracy degree raised, together with the need to obtain a more accurate solution to the relative orbital motion problem [7]. Recently, Gim and Alfriend [8] used the state transition matrix in the study of the relative motion. The main goal was to express the linearized equations of motion with respect to the initial conditions, with applications in formation initialization and reconfiguration.

Attempts to offer more accurate equations of motion starting from the nonlinear initial value problem that models the motion were made. Recently, Gurfil and Kasdin [9] derived closed-form expression of the relative position vector, but only when the reference trajectory is circular. Similar expressions for the law of relative motion starting from the nonlinear model are presented in [7,10–12].

The relative orbital motion problem was also studied from the point of view of the associated differential manifold. Gurfil and Kholshevnikov [13] introduced a metric which helps to study the relative distance between Keplerian orbits. Gronchi [14,15] also introduced a metric between two confocal Keplerian orbits and used this instrument in problems of asteroid and comet collisions.

In 2007, the authors of the present paper [16,17] offered the closed-form solution to the nonlinear unperturbed model of the relative orbital motion. The method led to closed-form vectorial coordinate-free expressions for the relative law of motion and relative velocity and it was based on an approach first introduced in 1995 [18]. It involves the Lie group of proper orthogonal tensor functions and its associated Lie algebra of skew-symmetric tensor functions. Then, the solution was generalized to the problem of the relative motion in a central force field [19–21].

An inedite solution to the Kepler problem by using the algebra of hypercomplex numbers was offered by the authors of the present paper [22]. Based on this solution and by using the hypercomplex eccentric anomaly, a unified closed-form solution to the relative orbital motion was determined [23].

The present approach offers a quaternionic procedure to obtain exact expressions for the relative law of motion and the relative velocity between two Keplerian confocal orbits. The solution is obtained by pure analytical methods and it holds for any chief and deputy trajectories, without involving any secular terms or singularities. The relative orbital motion is reduced, by an adequate change of variables, into the classic Kepler problem. It is proved that the relative orbital motion problem is superintegrable. The quaternions play only a catalyst role, the final solution being expressed in a vectorial form.

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The main result of this paper is the closed-form general solution for the relative orbital motion (in the unperturbed situation). This solution is obtained by using an adequate quaternionic change of variable in the nonlinear initial value problem, which governs the motion. To obtain this solution, one has to know only the inertial motion of the chief spacecraft and the initial conditions (position and velocity) of the deputy satellite in the local-vertical-local-horizontal (LVLH) frame. Both the relative law of motion and the relative velocity of the deputy are obtained, by using the quaternion instrument that is developed in the first part of the paper.

Another contribution of the authors is the expression of the solution to the relative orbital motion by using universal functions, in a compact and unified form. Another expression of the unique solution to the relative orbital motion problem may be found in [23], by using hypercomplex numbers.

The use of quaternion-valued functions instead of tensor-valued functions is preferred, because from the computational point of view the quaternion expressions are easier to handle. Also, quaternions are a more familiar term than tensors.

Sec. II contains an overview on the algebra of quaternions, useful for the present approach. Sec. III presents the quaternionic formulation of the relative orbital motion. Sec. IV offers the closed-form solution to this problem. Exact quaternionic expressions for the relative law of motion and the relative velocity are deduced in Sec. V. A special attention is paid to the ellipse–ellipse situation, where the equations of motion are offered in a closed-form exact expression with respect to the eccentric anomalies of the two satellites.

The Appendix offers the scalar equations which are obtained for the law of motion and the velocity from the general solution presented in this paper.

II. Mathematical Preliminaries

A. Algebra of Quaternions

Quaternions were introduced by the Irish mathematician Sir William Rowan Hamilton in 1843. He defined a quaternion as an expression of the form

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1}$$

where w, x, y, z are called the *constituents* of the quaternion and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are called *imaginary units* [24]. The imaginary units satisfy the well-known equalities

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1; \qquad \mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k} \\ \mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j} = \mathbf{i}; \qquad \mathbf{k}\mathbf{i} = -\mathbf{i}\mathbf{k} = \mathbf{j} \end{aligned} \tag{2}$$

The first constituent w of a quaternion is named *real part*, and the last three constituents, x, y, z (in this order) form the vector part of quaternion $\mathbf{q}$. We shall use quaternions in order to model rotations. We shall use the "modern" form of quaternions, namely a pair between a real number and a three-dimensional vector:

$$\mathbf{q} = (a_0, \mathbf{a}) \tag{3}$$

where a_0 is a real number and $\mathbf{a}$ is a vector. Here a_0 is named the real part of $\mathbf{q}$ and $\mathbf{a}$ the vector part of $\mathbf{q}$. A vector quaternion is defined as a quaternion with zero real part.

The set $\mathbb{H}$ of quaternions is an noncommutative, associative four-dimensional division algebra with respect to the scalar multiplication, quaternionic sum, and quaternionic product, defined as [25]

$$\begin{cases} \lambda(a_0, \mathbf{a}) = (\lambda a_0, \lambda\mathbf{a}); \\ \mathbf{q}_1 = (a_0, \mathbf{a}); \mathbf{q}_2 = (b_0, \mathbf{b}); \\ \mathbf{q}_1 + \mathbf{q}_2 = (a_0 + b_0, \mathbf{a} + \mathbf{b}); \\ \mathbf{q}_1\mathbf{q}_2 = (a_0 b_0 - \mathbf{a}\cdot\mathbf{b}, a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a}\times\mathbf{b}) \end{cases} \tag{4}$$

where $\cdot$ represents the vector dot product and $\times$ the vector cross product. From the mathematical point of view, the quaternion algebra is important because, according to the Frobenius theorem, it is the only noncommutative finite dimensional division algebra.

Remember that an algebra is a vector space where an additional internal operation, usually named product, may be defined. The dimension of an algebra is the algebraic dimension of the above-mentioned vector space. A division algebra is one where the division operation is possible; i.e., for any elements $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$, there exist two unique elements $\mathbf{x}$ and $\mathbf{y}$ in the algebra, such as

$$\mathbf{a} = \mathbf{x}\mathbf{b}; \qquad \mathbf{a} = \mathbf{b}\mathbf{y} \tag{5}$$

For a quaternion $\mathbf{q}$ that has the form as in Eq. (3), its conjugate is denoted $\mathbf{q}^*$ and it is defined as $\mathbf{q}^* = (a_0, -\mathbf{a})$. The norm of the quaternion $\mathbf{q} = (a_0, \mathbf{a})$ is defined by

$$\|\mathbf{q}\| \overset{\text{def}}{=} \sqrt{\mathbf{q}\mathbf{q}^*} = \sqrt{a_0^2 + |\mathbf{a}|^2} \tag{6}$$

where $|\mathbf{a}|$ denotes the magnitude of vector $\mathbf{a}$. We will use the same letters to denote a vector and its corresponding vector quaternion, i.e., $\mathbf{r} = (0, \mathbf{r})$. Remember that the vector dot product and cross product may be expressed in a quaternionic way as

$$\begin{cases} \mathbf{a}\cdot\mathbf{b} = -\frac{1}{2}(\mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a}); \\ \mathbf{a}\times\mathbf{b} = \frac{1}{2}(\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}) \end{cases} \tag{7}$$

as it follows from Eq. (4).

Quaternions are used mostly in rigid body kinematics. The motion of a particle on a sphere with a constant radius is described with the help of time-depending quaternions in the following way:

$$\mathbf{r}(t) = \mathbf{q}(t)\mathbf{r}_0\mathbf{q}^*(t) \tag{8}$$

where $\mathbf{r} = \mathbf{r}(t)$ is the vector quaternion that models the motion, $\mathbf{r}_0$ is a constant vector quaternion, and $\mathbf{q}(t)$ is a time-depending quaternion that satisfies $\|\mathbf{q}(t)\| = 1$. The finite rotation with angle $\alpha \in [0, 2\pi)$ of some vector $\mathbf{r}_0$ around an axis that has the orientation modeled by the vector quaternion $\mathbf{u}$, $\|\mathbf{u}\| = 1$ is modeled by [25]

$$\mathbf{r} = \mathbf{q}(\mathbf{u}, \alpha)\mathbf{r}_0\mathbf{q}^*(\mathbf{u}, \alpha) \tag{9}$$

where $\mathbf{q}(\mathbf{u}, \alpha)$ is defined as

$$\mathbf{q}(\mathbf{u}, \alpha) = \left(\cos\frac{\alpha}{2}, \mathbf{u}\sin\frac{\alpha}{2}\right) \tag{10}$$

B. Darboux Equation

In rigid body kinematics, an important problem is how to describe the instantaneous rotation of a rigid body when its instantaneous angular velocity is given [26]. The classic approach leads to a Riccati differential equation, which was explicitly solved only in a particular situation, namely when the instantaneous angular velocity is constant [27,28].

The rotation with angular velocity $\boldsymbol{\omega}$ of a constant vector $\mathbf{r}_0$ is described by

$$\mathbf{r} = \mathbf{R}\mathbf{r}_0 \tag{11}$$

The instantaneous angular velocity vector $\boldsymbol{\omega}$ associated to the proper orthogonal valued function is defined by

$$\tilde{\boldsymbol{\omega}} = \dot{\mathbf{R}}\mathbf{R}^T \tag{12}$$

where the matrix $\tilde{\boldsymbol{\omega}}$ is related to vector $\boldsymbol{\omega}$ by

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{13}$$

and vector $\boldsymbol{\omega}$ is represented by Cartesian coordinates with respect to a right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3 \tag{14}$$

In the contemporary language, the solution to the Darboux equation offers the rotation matrix that models the rotation with a given instantaneous angular velocity $\boldsymbol{\omega}$ (which must be a continuous vector function). The matrix formulation of the Darboux equation is

$$\dot{\mathbf{R}} = \tilde{\omega}\mathbf{R}, \qquad \mathbf{R}(t_0) = \mathbf{I}_3 \tag{15}$$

where $t_0 \geq 0$ denotes the initial moment of time and $\mathbf{R}$ is a 3×3 matrix whose elements are differentiable scalar functions.

The rotation matrix $\mathbf{R}$ associated with vector $\boldsymbol{\omega}$ is the solution to the initial value problem (15), and it is a proper orthogonal matrix function, i.e.,

$$\mathbf{R}\mathbf{R}^T = \mathbf{I}_3, \qquad \det \mathbf{R} = 1 \tag{16}$$

When using quaternions in order to describe rotations, let us consider $\boldsymbol{\omega}$ the vector quaternion corresponding to the instantaneous angular velocity vector and $\mathbf{q}$ the unit quaternion that models the rotation. The quaternionic operator defined as [29]

$$\mathbf{R}_{\omega} \stackrel{\text{def}}{=} \mathbf{q}(\,)\mathbf{q}^* \tag{17}$$

rotates any constant vector $\mathbf{r}_0$ with instantaneous angular velocity $\boldsymbol{\omega}$. From Eq. (12) it follows that

$$\tilde{\omega}\mathbf{r}_0 = \dot{\mathbf{R}}_{\omega}(\mathbf{R}_{\omega}^{-1}\mathbf{r}_0) \tag{18}$$

and by using vector quaternions we may rewrite Eq. (18) as

$$\frac{1}{2}(\boldsymbol{\omega}\mathbf{r}_0 - \mathbf{r}_0\boldsymbol{\omega}) = \dot{\mathbf{q}}(\mathbf{q}^*\mathbf{r}_0\mathbf{q})\mathbf{q}^* + \mathbf{q}(\mathbf{q}^*\mathbf{r}_0\mathbf{q})\dot{\mathbf{q}}^* = \dot{\mathbf{q}}\mathbf{q}^*\mathbf{r}_0 + \mathbf{r}_0\mathbf{q}\dot{\mathbf{q}}^* \tag{19}$$

Because $\mathbf{r}_0$ is an arbitrary constant vector quaternion, from Eq. (19) we deduce that the unit quaternion $\mathbf{q}$ that describes the rotation with angular velocity $\boldsymbol{\omega}$ is the solution to the Darboux-like equation

$$\dot{\mathbf{q}} = \frac{\boldsymbol{\omega}}{2}\mathbf{q}, \qquad \mathbf{q}(t_0) = \hat{\mathbf{1}} \tag{20}$$

where $\hat{\mathbf{1}} = (1, 0)$. Here, $\boldsymbol{\omega}$ is the vector quaternion associated with vector $\boldsymbol{\omega}$ from Eq. (15).

In the present approach, a greater attention is paid to the rotation with angular velocity $-\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is a continuous vector quaternion function. By taking into account Eq. (15) and the expression of $\mathbf{R}_{\omega}$ in Eq. (17), it follows that the continuous rotation with instantaneous angular velocity modeled by the vector quaternion $-\boldsymbol{\omega}$ is the solution to the quaternionic initial value problem:

$$\dot{\mathbf{q}} + \frac{\boldsymbol{\omega}}{2}\mathbf{q} = \mathbf{0}, \qquad \mathbf{q}(t_0) = \hat{\mathbf{1}} \tag{21}$$

Generally, the unique solution of Eq. (21) may not be explicitly expressed with respect to $\boldsymbol{\omega}$. There exists a numerical solution of the initial value problem (21) when the instantaneous angular velocity $\boldsymbol{\omega}$ is an arbitrary continuous vector quaternion function. This is known as the Peano–Baker solution; it is obtained by iteration [27,30], and it is presented as a limit of infinitesimal integrals in the general situation. The initial value problem (21) is equivalent to the integral quaternionic equation:

$$\mathbf{q}(t) = \hat{\mathbf{1}} - \frac{1}{2}\int_{t_0}^{t} \boldsymbol{\omega}(\tau)\mathbf{q}(\tau)\,\mathrm{d}\tau \tag{22}$$

Equation (22) has the solution

$$\mathbf{q}(t) = \hat{\mathbf{1}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \int_{t_0}^{t} \mathrm{d}t_1 \boldsymbol{\omega}(t_1) \int_{t_0}^{t_1} \mathrm{d}t_2 \boldsymbol{\omega}(t_2) \ldots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n \boldsymbol{\omega}(t_n) \tag{23}$$

which is known as the Peano–Baker solution [30]. A concise formulation of Eq. (23) is obtained by using the following procedure [27]: because the order of multiplication is important [31] and generally $\boldsymbol{\omega}(t_k)\boldsymbol{\omega}(t_p) \neq \boldsymbol{\omega}(t_p)\boldsymbol{\omega}(t_k)$, $k, p = \overline{1, n}$, the function

$$\theta(t) = \begin{cases} 0, & t \leq t_0 \\ 1, & t > t_0 \end{cases} \tag{24}$$

is introduced. By considering $\mathcal{P}(n)$ the group of permutations of the set $\{1, \ldots, n\}$, one may define the time-ordered product of the vector quaternions $\boldsymbol{\omega}(t_1), \ldots, \boldsymbol{\omega}(t_n)$ as

$$\mathbf{T}[\boldsymbol{\omega}(t_1), \ldots, \boldsymbol{\omega}(t_n)] \stackrel{\text{def}}{=} \frac{(-1)^n}{2^n} \sum_{\sigma \in \mathcal{P}(n)} \left[\prod_{k=1}^{n-1} \theta(t_{\sigma(k)} - t_{\sigma(k+1)}) \prod_{p=1}^{n} \boldsymbol{\omega}(t_{\sigma(p)})\right] \tag{25}$$

Now, the solution (23) of the initial value problem (22) becomes

$$\mathbf{q}(t) = \hat{\mathbf{1}} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^{t} \mathrm{d}t_1 \ldots \int_{t_0}^{t} \mathrm{d}t_n \mathbf{T}[\boldsymbol{\omega}(t_1), \ldots, \boldsymbol{\omega}(t_n)] \tag{26}$$

If the vector part of the vector quaternion $\boldsymbol{\omega}$ has a fixed direction, the solution to the initial value problem (21) may be written as

$$\mathbf{q}(t) = \exp\left[-\frac{1}{2}\int_{t_0}^{t} \boldsymbol{\omega}(\tau)\,\mathrm{d}\tau\right] \tag{27}$$

If the direction of the instantaneous angular velocity is fixed, one may consider the vector quaternion $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ as

$$\boldsymbol{\omega}(t) = \omega(t)\mathbf{u} \tag{28}$$

where $\omega = \omega(t)$ is a continuous scalar function and $\mathbf{u}$ is a vector quaternion corresponding to the unit vector that gives the orientation of the instantaneous angular velocity. By denoting

$$\varphi(t) = \int_{t_0}^{t} \omega(\tau)\,\mathrm{d}\tau \tag{29}$$

and making the computations, the explicit solution to the quaternionic form (21) of the Darboux equation arises from Eq. (27):

$$\mathbf{q}(t) = \left(\cos\frac{\varphi(t)}{2}, -\mathbf{u}\sin\frac{\varphi(t)}{2}\right) \tag{30}$$

In the present approach, the explicit form (30) of the solution to the quaternionic Darboux Eq. (21) is fundamental. It will allow the determination of closed-form exact expressions for the vector quaternions that model the relative position and the relative velocity in the relative orbital motion problem.

C. Quaternionic Operator

This section presents a useful instrument that is fundamental in determining the solution to the relative orbital motion problem. Consider the quaternionic operator $\mathbf{F}_{\boldsymbol{\omega}}$ defined as

$$\mathbf{F}_{\boldsymbol{\omega}} \overset{\text{def}}{=} \mathbf{q}^*(\,)\mathbf{q} \tag{31}$$

where $\mathbf{q}$ is the solution to the initial value problem:

$$\dot{\mathbf{q}} = -\frac{\boldsymbol{\omega}}{2}\mathbf{q}, \qquad \mathbf{q}(t_0) = \hat{\mathbf{1}} \tag{32}$$

In this approach, the direction of the vector associated to the vector quaternion $\boldsymbol{\omega}$ is assumed to be fixed. The following properties arise from Eqs. (31) and (32):

1) $\mathbf{F}_{\boldsymbol{\omega}}$ is a linear operator; i.e., for any two quaternions $\mathbf{a}$, $\mathbf{b}$ and for any two scalars $\lambda_{1,2}$,

$$\mathbf{F}_{\boldsymbol{\omega}}(\lambda_1\mathbf{a} + \lambda_2\mathbf{b}) = \lambda_1\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a} + \lambda_2\mathbf{F}_{\boldsymbol{\omega}}\mathbf{b} \tag{33}$$

2) $\mathbf{F}_{\boldsymbol{\omega}}$ preserves the quaternionic product [defined in the last equation of Eq. (4)]; i.e., for any two quaternions $\mathbf{a}$, $\mathbf{b}$,

$$\mathbf{F}_{\boldsymbol{\omega}}(\mathbf{ab}) = (\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a})(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{b}) \tag{34}$$

3) $\mathbf{F}_{\boldsymbol{\omega}}$ preserves the quaternionic norm; i.e., for any quaternion $\mathbf{a}$,

$$\|\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}\| = \|\mathbf{a}\| \tag{35}$$

4) The derivative with respect to time of $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}$, where $\mathbf{a}$ is an arbitrary quaternion-valued function of real variable t, is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}) = \mathbf{F}_{\boldsymbol{\omega}}\left[\dot{\mathbf{a}} + \frac{1}{2}(\boldsymbol{\omega}\mathbf{a} - \mathbf{a}\boldsymbol{\omega})\right] \tag{36}$$

5) The second derivative with respect to time of $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}$, where $\mathbf{a}$ is an arbitrary quaternion-valued function of real variable t, is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}) = \mathbf{F}_{\boldsymbol{\omega}}\left[\ddot{\mathbf{a}} + (\boldsymbol{\omega}\dot{\mathbf{a}} - \dot{\mathbf{a}}\boldsymbol{\omega}) + \frac{1}{4}(\boldsymbol{\omega}^2\mathbf{a} + \mathbf{a}\boldsymbol{\omega}^2 - 2\boldsymbol{\omega}\mathbf{a}\boldsymbol{\omega}) + \frac{1}{2}(\dot{\boldsymbol{\omega}}\mathbf{a} - \mathbf{a}\dot{\boldsymbol{\omega}})\right] \tag{37}$$

6) $\mathbf{F}_{\boldsymbol{\omega}}$ is invertible and its inverse, which we denote $\mathbf{R}_{-\boldsymbol{\omega}}$, is

$$\mathbf{R}_{-\boldsymbol{\omega}} = (\mathbf{F}_{\boldsymbol{\omega}})^{-1} = \mathbf{q}(\,)\mathbf{q}^* \tag{38}$$

and it models the rotation with the instantaneous angular velocity modeled by the vector quaternion $-\boldsymbol{\omega}$. One may recall that $\mathbf{q}$ is defined by the initial value problem (32).

The above properties of the quaternion operator $\mathbf{F}_{\boldsymbol{\omega}}$ are essential in solving the relative orbital motion problem in the unperturbed case.

III. Problem Formulation

Consider two spacecraft orbiting the same attraction center. One will be considered as the reference spacecraft and will be named *chief*, and the other will be named *deputy*. Assume that only the gravitational attraction force determines the trajectories of both spacecraft. In this situation, their trajectories may be (independently) elliptic, parabolic, hyperbolic, or rectilinear. The relative orbital motion problem is to express the motion of the deputy spacecraft with respect to a reference frame that is originated in the mass center C of the chief spacecraft and has the axes oriented as follows: the X axis has the same orientation as the position vector of the chief with respect to the attraction center, the Z axis has the same orientation as the angular momentum $\mathbf{h}_C$ of the chief orbit, and the Y axis completes a right-oriented orthogonal frame $CXYZ$, as shown in Fig. 1. This frame [25] is usually named LVLH. The input data contains all informations related to the inertial state (position and velocity) of the chief spacecraft at any moment of time and the relative position and the relative velocity of the deputy spacecraft with respect to LVLH at the initial moment of time.

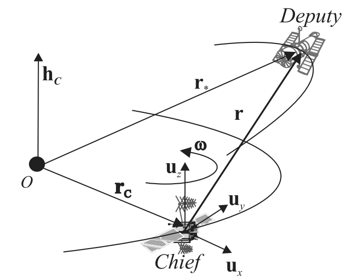

Fig. 1 Physical model of the relative orbital motion.

The LVLH frame is noninertial. It rotates with instantaneous angular velocity $\boldsymbol{\omega}$, and the inertial acceleration of point C is

$$\mathbf{a}_C = -\mu\left[\frac{1 + e_C\cos f_C(t)}{p_C}\right]^2\mathbf{u}_x \tag{39}$$

Consider the vector quaternion $\mathbf{r}_C$ such as $-\mathbf{r}_C$ models the position of the attraction center with respect to LVLH.

Then, $\mathbf{r}_C$ models a rectilinear motion, and its expression is

$$\mathbf{r}_C = \frac{p_C}{1 + e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{40}$$

Here, e_C denotes the eccentricity of the Keplerian chief trajectory, $f_C = f_C(t)$ denotes the chief true anomaly, and p_C represents the semilatus rectum of the chief trajectory. Here, $\mathbf{u}_x$ is the vector quaternion associated to the unit vector that has the same orientation as the X axis of the LVLH frame.

The vector quaternion $\mathbf{r}_C$ is the solution to the initial value problem:

$$\ddot{\mathbf{r}}_C + (\boldsymbol{\omega}\dot{\mathbf{r}}_C - \dot{\mathbf{r}}_C\boldsymbol{\omega}) + \frac{1}{4}(\boldsymbol{\omega}^2\mathbf{r}_C + \mathbf{r}_C\boldsymbol{\omega}^2 - 2\boldsymbol{\omega}\mathbf{r}_C\boldsymbol{\omega}) + \frac{1}{2}(\dot{\boldsymbol{\omega}}\mathbf{r}_C - \mathbf{r}_C\dot{\boldsymbol{\omega}}) + \frac{\mu}{\|\mathbf{r}_C\|^3}\mathbf{r}_C = \mathbf{0}$$
$$\begin{cases}\mathbf{r}_C(t_0) = \mathbf{r}_C^0,\\ \dot{\mathbf{r}}_C(t_0) = (\mathbf{v}_C^0\cdot\mathbf{u}_x)\mathbf{u}_x\end{cases} \tag{41}$$

where $\mathbf{v}_C^0$ is the initial velocity of the chief spacecraft with respect to the inertial reference frame originated in the attraction center.

The vector $\boldsymbol{\omega}$ has the same direction as the chief angular momentum $\mathbf{h}_C$, and its expression is

$$\boldsymbol{\omega} = \left[\frac{1 + e_C\cos f_C(t)}{p_C}\right]^2\mathbf{h}_C \tag{42}$$

We denote by $\mathbf{r}$ the vector quaternion that models the position of the deputy with respect to the LVLH frame. The initial value problem that models the relative orbital motion has the quaternionic form

$$\ddot{\mathbf{r}} + (\boldsymbol{\omega}\dot{\mathbf{r}} - \dot{\mathbf{r}}\boldsymbol{\omega}) + \frac{1}{4}(\boldsymbol{\omega}^2\mathbf{r} + \mathbf{r}\boldsymbol{\omega}^2 - 2\boldsymbol{\omega}\mathbf{r}\boldsymbol{\omega}) + \frac{1}{2}(\dot{\boldsymbol{\omega}}\mathbf{r} - \mathbf{r}\dot{\boldsymbol{\omega}}) + \frac{\mu}{\|\mathbf{r} + \mathbf{r}_C\|^3}(\mathbf{r} + \mathbf{r}_C) - \frac{\mu}{\|\mathbf{r}_C\|^3}\mathbf{r}_C = \mathbf{0}$$
$$\begin{cases}\mathbf{r}(t_0) = \Delta\mathbf{r},\\ \dot{\mathbf{r}}(t_0) = \Delta\mathbf{v}\end{cases} \tag{43}$$

In Eq. (43), $\mu > 0$ represents the gravitational parameter of the attraction center, $-\mathbf{r}_C$ is the vector quaternion that models the position of the attraction center with respect to LVLH, $\Delta\mathbf{r}$ and $\Delta\mathbf{v}$ are vector quaternions that model the position and the velocity of the deputy with respect to LVLH at the initial moment of time $t_0 \geq 0$.

Equation (43) is a strong nonlinear quaternionic initial value problem with variable coefficients. We will offer the closed-form solution to Eq. (43) by using a quaternionic instrument that relates the motion in a rotating reference frame to the motion in an inertial reference frame. The solution is presented in a coordinate-free form, and it allows us to derive exact scalar expressions for both law of motion and velocity with respect to any particular coordinate system. The solution to the initial value problem (43) offers a parameterization of the differentiable manifold associated to the relative orbital motion [13]. This parameterization may be determined in any initial conditions $\Delta\mathbf{r}$ $\Delta\mathbf{v}$, of the initial value problem (43).

IV. Quaternionic Closed-Form Solution to the Relative Orbital Motion Problem

In this section we present the closed-form exact solution to Eq. (43). In the initial value problem (43), we make the change of variable

$$\mathbf{r}_* = \mathbf{F}_{\boldsymbol{\omega}}(\mathbf{r} + \mathbf{r}_C) \tag{44}$$

and recall that $\mathbf{r}_C$ is the solution to the initial value problem (41). After some algebra, it follows that

$$\ddot{\mathbf{r}}_* = \mathbf{F}_{\boldsymbol{\omega}}\left\{(\ddot{\mathbf{r}} + \ddot{\mathbf{r}}_C) + [\boldsymbol{\omega}(\dot{\mathbf{r}} + \dot{\mathbf{r}}_C) - (\dot{\mathbf{r}} + \dot{\mathbf{r}}_C)\boldsymbol{\omega}] + \frac{1}{4}[\boldsymbol{\omega}^2(\mathbf{r} + \mathbf{r}_C) + (\mathbf{r} + \mathbf{r}_C)\boldsymbol{\omega}^2 - 2\boldsymbol{\omega}(\mathbf{r} + \mathbf{r}_C)\boldsymbol{\omega}] + \frac{1}{2}[\dot{\boldsymbol{\omega}}(\mathbf{r} + \mathbf{r}_C) - (\mathbf{r} - \mathbf{r}_C)\dot{\boldsymbol{\omega}}]\right\} \tag{45}$$

and furthermore

$$\ddot{\mathbf{r}}_* = \mathbf{F}_{\boldsymbol{\omega}}\left\{\ddot{\mathbf{r}} + [\boldsymbol{\omega}\dot{\mathbf{r}} - \dot{\mathbf{r}}\boldsymbol{\omega}] + \frac{1}{4}[\boldsymbol{\omega}^2\mathbf{r} + \mathbf{r}\boldsymbol{\omega}^2 - 2\boldsymbol{\omega}\mathbf{r}\boldsymbol{\omega}] + \frac{1}{2}[\dot{\boldsymbol{\omega}}\mathbf{r} - \mathbf{r}\dot{\boldsymbol{\omega}}]\right\} + \mathbf{F}_{\boldsymbol{\omega}}\left\{\ddot{\mathbf{r}}_C + [\boldsymbol{\omega}\dot{\mathbf{r}}_C - \dot{\mathbf{r}}_C\boldsymbol{\omega}] + \frac{1}{4}[\boldsymbol{\omega}^2\mathbf{r}_C + \mathbf{r}_C\boldsymbol{\omega}^2 - 2\boldsymbol{\omega}\mathbf{r}_C\boldsymbol{\omega}] + \frac{1}{2}[\dot{\boldsymbol{\omega}}\mathbf{r}_C - \mathbf{r}_C\dot{\boldsymbol{\omega}}]\right\} \tag{46}$$

By using Eqs. (41) and (43), we obtain

$$\ddot{\mathbf{r}}_* = \mathbf{F}_{\boldsymbol{\omega}}\left[\frac{\mu}{\|\mathbf{r}_C\|^3}\mathbf{r}_C - \frac{\mu}{\|\mathbf{r} + \mathbf{r}_C\|^3}(\mathbf{r} + \mathbf{r}_C) - \frac{\mu}{\|\mathbf{r}_C\|^3}\mathbf{r}_C\right] = -\frac{\mu}{\|\mathbf{r} + \mathbf{r}_C\|^3}\mathbf{F}_{\boldsymbol{\omega}}(\mathbf{r} + \mathbf{r}_C) \tag{47}$$

In the end, from Eqs. (35) and (44), we obtain

$$\ddot{\mathbf{r}}_* + \frac{\mu}{\|\mathbf{r}_*\|^3}\mathbf{r}_* = \mathbf{0} \tag{48}$$

The initial conditions for the quaternionic differential Eq. (48) are deduced by taking into account that $\mathbf{F}_{\boldsymbol{\omega}}(t_0) = \hat{\mathbf{1}}$:

$$\mathbf{r}_*(t_0) = \mathbf{r}_C^0 + \Delta\mathbf{r} \tag{49}$$

$$\dot{\mathbf{r}}_*(t_0) = \mathbf{v}_C^0 + \Delta\mathbf{v} + \frac{1}{2}[\boldsymbol{\omega}(t_0)\Delta\mathbf{r} - \Delta\mathbf{r}\boldsymbol{\omega}(t_0)] \tag{50}$$

Having in mind Eqs. (38) and (44), we deduce

$$\mathbf{r} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_* - \mathbf{r}_C \tag{51}$$

The above considerations lead to the main result of this paper. This is stated thus: the solution to the relative orbital motion problem, described by the initial value problem (43), is

$$\mathbf{r} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_* - \frac{p_C}{1 + e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{52}$$

where $\mathbf{R}_{-\boldsymbol{\omega}}$ is defined in Eq. (38) and $\mathbf{r}_*$ is the solution to the initial value problem:

$$\ddot{\mathbf{r}}_* + \frac{\mu}{\|\mathbf{r}_*\|^3}\mathbf{r}_* = \mathbf{0}$$
$$\begin{cases}\mathbf{r}_*(t_0) = \mathbf{r}_C^0 + \Delta\mathbf{r}, \\ \dot{\mathbf{r}}_*(t_0) = \mathbf{v}_C^0 + \Delta\mathbf{v} + \frac{1}{2}[\boldsymbol{\omega}(t_0)\Delta\mathbf{r} - \Delta\mathbf{r}\boldsymbol{\omega}(t_0)]\end{cases} \tag{53}$$

and the relative velocity may be computed as

$$\mathbf{v} = \mathbf{R}_{-\boldsymbol{\omega}}\dot{\mathbf{r}}_* - \tilde{\boldsymbol{\omega}}\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_* - \frac{e_C\|\mathbf{h}_C\|\sin f_C}{p_C}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{54}$$

where we used the notation

$$\tilde{\boldsymbol{\omega}}\mathbf{x} = \frac{1}{2}(\boldsymbol{\omega}\mathbf{x} - \mathbf{x}\boldsymbol{\omega}) \tag{55}$$

for any quaternion $\mathbf{x}$.

This result shows a very interesting property of the relative orbital motion problem. We have proven that this problem is super-integrable, by reducing it to the classic Kepler problem. The solution of the relative orbital motion problem is expressed thus:

$$\mathbf{r} = \mathbf{r}(t, t_0, \Delta\mathbf{r}, \Delta\mathbf{v}); \qquad \mathbf{v} = \mathbf{v}(t, t_0, \Delta\mathbf{r}, \Delta\mathbf{v}) \tag{56}$$

Now, let $\mathbf{h}$ be the vector quaternion associated to the specific angular momentum of the Keplerian motion described by the initial value problem (53):

$$\mathbf{h} = (\mathbf{r}_C^0 + \Delta\mathbf{r}) \times \left[\mathbf{v}_C^0 + \Delta\mathbf{v} + \frac{1}{2}[\boldsymbol{\omega}(t_0)\Delta\mathbf{r} - \Delta\mathbf{r}\boldsymbol{\omega}(t_0)]\right] \tag{57}$$

and denote

$$\tilde{\mathbf{u}}_C = \frac{\tilde{\mathbf{h}}_C}{\|\mathbf{h}_C\|}; \qquad \tilde{\mathbf{u}} = \frac{\tilde{\mathbf{h}}}{\|\mathbf{h}\|}; \qquad r_* = \frac{p}{1 + e\cos f(t)} \tag{58}$$

Note that the quaternion operator $\mathbf{R}_{-\boldsymbol{\omega}}$ has an explicit expression, because the direction of the instantaneous angular velocity of the reference frame remains fixed. We may write

$$\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_* = \left(\cos\frac{f_C^0(t)}{2}, -\mathbf{u}_C\sin\frac{f_C^0(t)}{2}\right)\mathbf{r}_*\left(\cos\frac{f_C^0(t)}{2}, \mathbf{u}_C\sin\frac{f_C^0(t)}{2}\right) \tag{59}$$

where

$$f_C^0(t) = f_C(t) - f_C(t_0) \tag{60}$$

and $\mathbf{u}$ is the constant unit vector quaternion that models the fixed direction of $\boldsymbol{\omega}$.

By taking into account the closed-form expression of the quaternionic operator $\mathbf{R}_{-\boldsymbol{\omega}}$, it follows that the general vectorial expressions for the law of motion and velocity in the relative orbital motion are

$$\mathbf{r} = \frac{1}{\|\mathbf{h}_C\|^2}(\mathbf{h}_C\cdot\mathbf{r}_*)\mathbf{h}_C - \frac{1}{\|\mathbf{h}_C\|}\sin f_C^0(t)(\tilde{\mathbf{h}}_C\mathbf{r}_*) - \frac{1}{\|\mathbf{h}_C\|^2}\cos f_C^0(t)(\tilde{\mathbf{h}}_C\mathbf{r}_*) - \frac{p_C}{1 + e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{61}$$

$$\mathbf{v} = \frac{1}{\|\mathbf{h}_C\|^2}(\mathbf{h}_C \cdot \dot{\mathbf{r}}_*)\mathbf{h}_C - \frac{1}{\|\mathbf{h}_C\|}\sin f_C^0(t)(\tilde{\mathbf{h}}_C\dot{\mathbf{r}}_*) - \frac{1}{\|\mathbf{h}_C\|^2}\cos f_C^0(t)(\tilde{\mathbf{h}}_C^2\dot{\mathbf{r}}_*) - \left(\frac{1+e_C\cos f_C(t)}{p_C}\right)^2 \cos f_C^0(t)(\tilde{\mathbf{h}}_C\mathbf{r}_*) + \frac{1}{\|\mathbf{h}_C\|}\left(\frac{1+e_C\cos f_C(t)}{p_C}\right)^2 \sin f_C^0(t)(\tilde{\mathbf{h}}_C^2\mathbf{r}_*) - \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{62}$$

The solution (52) to the relative orbital motion problem may still be refined to another elegant formulation. Since the solution to the initial value problem (53) models a classic inertial Keplerian motion, the direction of vector $\mathbf{r}_*$ rotates with an angular velocity, which also has a constant direction, that of the vector quaternion $\mathbf{r}_*(t_0) \times \dot{\mathbf{r}}_*(t_0)$, so the vector quaternion $\mathbf{r}_*$ may be expressed as

$$\mathbf{r}_* = \frac{p}{1+e\cos f(t)}\mathbf{R}_{\boldsymbol{\alpha}}\frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} \tag{63}$$

where p is the semilatus rectum, e is the eccentricity, and $f(t)$ is the true anomaly, all associated to the Keplerian motion described by Eq. (53). The quaternion operator $\mathbf{R}_{\boldsymbol{\alpha}}$ has the exact expression

$$\mathbf{R}_{\boldsymbol{\alpha}}() = \left(\cos\frac{f^0(t)}{2}, \mathbf{u}\sin\frac{f^0(t)}{2}\right)()\left(\cos\frac{f^0(t)}{2}, -\mathbf{u}\sin\frac{f^0(t)}{2}\right) \tag{64}$$

where $\mathbf{u}$ is defined in (58) and

$$f^0(t) = f(t) - f(t_0) \tag{65}$$

So we may now offer another closed form of the solution to the relative orbital motion problem, stated thus: the solution to the initial value problem (43) is

$$\mathbf{r} = \frac{p}{1+e\cos f(t)}\mathbf{R}_*\frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} - \frac{p_C}{1+e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{66}$$

where $\mathbf{R}_*$ is defined as

$$\mathbf{R}_* = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{R}_{-\boldsymbol{\alpha}} \tag{67}$$

and the relative velocity may be computed as

$$\mathbf{v} = \frac{e\|\mathbf{h}\|\sin f(t)}{p}\mathbf{R}_*\frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} + \frac{p}{1+e\cos f(t)}\tilde{\boldsymbol{\omega}}_*\left(\mathbf{R}_*\frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|}\right) - \frac{e_C\|\mathbf{h}_C\|\sin f_C}{p_C}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{68}$$

where $\boldsymbol{\omega}_*$ is the vector quaternion that models the instantaneous angular velocity associated to the rotation described by $\mathbf{R}_*$; its expression is

$$\boldsymbol{\omega}_* = -\boldsymbol{\omega} + \frac{1}{\|\mathbf{r}_*\|^2}\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{h} \tag{69}$$

From Eqs. (66) and (68), we obtain another expression of the vectorial closed-form solution to the relative orbital motion problem:

$$\mathbf{r} = \frac{p}{1+e\cos f(t)}[\mathbf{I}_3 + \sin f^0(t)\tilde{\mathbf{u}} + (1-\cos f^0(t))\tilde{\mathbf{u}}^2 + (1-\cos f^0(t))\tilde{\mathbf{u}}_C^2 - \sin f_C^0\tilde{\mathbf{u}}_C - \sin f_C^0(t)\sin f^0(t)\tilde{\mathbf{u}}_C\tilde{\mathbf{u}} - \sin f_C^0(t)(1-\cos f^0(t))\tilde{\mathbf{u}}_C\tilde{\mathbf{u}}^2 + (1-\cos f_C^0(t))\sin f^0(t)\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}} + (1-\cos f_C^0(t))(1-\cos f^0(t))\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}}^2]\frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} - \frac{p_C}{1+e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{70}$$

$$\begin{aligned}\mathbf{v} = {}& \frac{e\|\mathbf{h}\|\sin f(t)}{p}[\mathbf{I}_3 + \sin f^0(t)\tilde{\mathbf{u}} + (1-\cos f^0(t))\tilde{\mathbf{u}}^2 + (1-\cos f^0(t))\tilde{\mathbf{u}}_C^2 - \sin f_C^0\tilde{\mathbf{u}}_C^2 - \sin f_C^0(t)\sin f^0(t)\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}}^2 \\ & - \sin f_C^0(t)(1-\cos f^0(t))\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}}^2 + (1-\cos f_C^0(t))\sin f^0(t)\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}} + (1-\cos f_C^0(t))(1-\cos f^0(t))\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}}^2]\frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} \\ & + r_*\Bigg[\frac{1}{r_*^2}\cos f^0(t)\tilde{\mathbf{h}} + \frac{\|\mathbf{h}\|}{r_*^2}\sin f^0(t)\tilde{\mathbf{u}}^2 + \frac{\|\mathbf{h}\|}{r_*^2}\sin f^0(t)\tilde{\mathbf{u}}_C^2 - \frac{1}{r_C^2}\cos f_C^0(t)\tilde{\mathbf{h}}_C^2 - \left(\frac{\|\mathbf{h}\|}{r_*^2}\sin f_C^0(t)\cos f^0(t) + \frac{\|\mathbf{h}_C\|}{r_C^2}\cos f_C^0(t)\sin f^0(t)\right)\tilde{\mathbf{u}}_C\tilde{\mathbf{u}} \\ & - \frac{1}{r_C^2}\cos f_C^0(t)(1-\cos f^0(t))\tilde{\mathbf{h}}_C\tilde{\mathbf{u}}^2 + \frac{\|\mathbf{h}\|}{r_*^2}\sin f_C^0(t)\sin f^0(t)\tilde{\mathbf{u}}_C\tilde{\mathbf{u}}^2 + \frac{\|\mathbf{h}_C\|}{r_C^2}\sin f_C^0(t)\sin f^0(t)\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}} - \frac{1}{r_*^2}(1-\cos f_C^0(t))\cos f^0(t)\tilde{\mathbf{u}}_C^2\tilde{\mathbf{h}} \\ & + \frac{\|\mathbf{h}_C\|}{r_C^2}\sin f_C^0(t)(1-\cos f^0(t))\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}}^2 + \frac{\|\mathbf{h}\|}{r_*^2}(1-\cos f_C^0(t))\sin f^0(t)\tilde{\mathbf{u}}_C^2\tilde{\mathbf{u}}^2\Bigg]\frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} - \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|}\end{aligned} \tag{71}$$

Here, we present another formulation of the solution to the relative orbital motion. Let U_k, $k = \overline{\mathbf{0},\mathbf{3}}$, $U_k = U_k(\chi;\alpha)$, be the universal functions defined in [25], pp. 175–179, with

$$\alpha = \frac{2}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} - \frac{\|\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}\|^2}{\mu} = -\mu\xi \tag{72}$$

and χ a Sundman-like independent universal variable that satisfies

$$\frac{dt}{d\chi} = \frac{1}{\sqrt{\mu}}r_* \tag{73}$$

Then, the solution to the initial value problem (57) may be expressed as [25]

$$\mathbf{r}_* = \left\{U_0 + \left[\frac{1}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} - \frac{\|\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}\|^2}{\mu}\right]U_2\right\}(\mathbf{r}_C^0 + \Delta\mathbf{r}) + \left[U_1 \frac{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|}{\sqrt{\mu}} + U_2 \frac{(\mathbf{r}_C^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}) \tag{74}$$

and the magnitude of the solution is [25]

$$r_* = \|\mathbf{r}_C^0 + \Delta\mathbf{r}\|U_0 + \frac{(\mathbf{r}_C^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu} U_1 + U_2 \tag{75}$$

The velocity of the motion governed by Eq. (53) is

$$\dot{\mathbf{r}}_* = -\frac{\sqrt{\mu}}{r_*} U_1 \frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} + \frac{\sqrt{\mu}}{r_*}\left[U_0 \frac{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|}{\sqrt{\mu}} + U_1 \frac{(\mathbf{r}_C^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}) \tag{76}$$

Then, the solution to the initial value problem (43) may be written as

$$\mathbf{r} = \mathbf{R}_{-\omega}\left\{\left\{U_0 + \left[\frac{1}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} - \frac{\|\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}\|^2}{\mu}\right]U_2\right\}(\mathbf{r}_C^0 + \Delta\mathbf{r}) + \left[U_1 \frac{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|}{\sqrt{\mu}} + U_2 \frac{(\mathbf{r}_C^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\right\} - \frac{p_C}{1 + e_C \cos f_C(t)} \frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{77}$$

$$\mathbf{v} = \mathbf{R}_{-\omega}\left\{-\frac{\sqrt{\mu}}{r_*} U_1 \frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} + \frac{\sqrt{\mu}}{r_*}\left[U_0 \frac{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|}{\sqrt{\mu}} + U_1 \frac{(\mathbf{r}_C^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\right\} - \tilde{\boldsymbol{\omega}}\mathbf{R}_{-\omega}\left\{\left\{U_0 + \left[\frac{1}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} - \frac{\|\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}\|^2}{\mu}\right]U_2\right\}(\mathbf{r}_C^0 + \Delta\mathbf{r}) + \left[U_1 \frac{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|}{\sqrt{\mu}} + U_2 \frac{(\mathbf{r}_C^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\mu}\right] \times (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})\right\} - \frac{e_C\|\mathbf{h}_C\| \sin f_C}{p_C} \frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{78}$$

The universal functions U_k are linked by a Kepler-like equation [25]:

$$\sqrt{\mu}(t - t_0) = \|\mathbf{r}_C^0 + \Delta\mathbf{r}\|U_1(\chi;\alpha) + \frac{(\mathbf{r}_C^0 + \Delta\mathbf{r}) \cdot (\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r})}{\sqrt{\mu}} U_2(\chi;\alpha) + U_3(\chi;\alpha) \tag{79}$$

Equations (77) and (78) offer the closed-form compact solution to the relative orbital motion problem. They hold for any chief or deputy trajectories.

V. Comprehensive Analysis of the Relative Orbital Motion

By using the results presented in the previous sections of this paper, we are about to offer the closed-form solution to the relative orbital motion in all possible particular cases. In this approach, the chief inertial trajectory is less important than the deputy inertial trajectory, and the study will focus on the nature of the latter. We must make here the remark that in fact the initial value problem (53) models the motion of the deputy spacecraft in the inertial frame. This equation is deduced by knowing only the chief motion and the initial conditions of the deputy in the LVLH frame. From this point, when referring to the deputy inertial motion, we refer in fact to the motion governed by the initial value problem (53).

We offer the closed-form solution to the nonlinear model of the relative orbital motion in the situation where the inertial deputy trajectory is an ellipse, a parabola, or a hyperbola. These situations are delimited by the sign of the generalized specific energy of the deputy spacecraft [17,32,33]. It was proven that in the conditions that are given above, the sign of the quantity

$$\xi = \frac{1}{2}\|[\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}]\|^2 - \frac{\mu}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} \tag{80}$$

gives the type of the Keplerian inertial trajectory of the deputy spacecraft, i.e., if $\xi < 0$ the inertial trajectory of the deputy is an ellipse, if $\xi = 0$ it is a parabola, and if $\xi > 0$ it is a hyperbola. An accurate observer would remark that the previous phrase is mathematically correct only if the angular momentum $\mathbf{h}$ of the deputy inertial orbit is nonzero, $\mathbf{h} \neq \hat{\mathbf{0}}$. Only this situation will be taken into consideration in this approach.

All computations will start from the solution presented in Eqs. (61) and (62).

A. Elliptic Inertial Deputy Trajectory ($\xi < 0, \mathbf{h} \neq \mathbf{0}$)

The inertial trajectory of the deputy spacecraft is an ellipse (or a circle). The motion on this orbit is modeled by the position vector $\mathbf{r}_*$, which is the solution to the initial value problem (53). The expressions for the quaternion vectors $\mathbf{r}_*$ and $\dot{\mathbf{r}}_*$ are [34]

$$\mathbf{r}_* = \mathbf{a}[\cos E(t) - e] + \mathbf{b}\sin E(t) \tag{81}$$

$$\dot{\mathbf{r}}_* \frac{n}{1 - e\cos E(t)}[-\mathbf{a}\sin E(t) + \mathbf{b}\cos E(t)] \tag{82}$$

where $\mathbf{e}$ represents the vector quaternion corresponding to the vectorial eccentricity of the Keplerian motion described by Eq. (53); its expression is

$$\mathbf{e} = \frac{1}{\mu}(\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}) \times \mathbf{h} - \frac{\mathbf{r}_C^0 + \Delta\mathbf{r}}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|} \tag{83}$$

if $\mathbf{e} = \mathbf{0}$, the inertial trajectory of the deputy spacecraft is circular; $\mathbf{h}$ is defined in Eq. (57); n is the mean motion of the motion described by Eq. (53); $\mathbf{a}$ and $\mathbf{b}$ represent the vectors that model the semimajor and semiminor axis of the deputy inertial trajectory respectively; their expressions are

$$n = \frac{(2|\xi|)^{\frac{3}{2}}}{\mu}; \qquad \mathbf{a} = \begin{cases} \frac{\mu}{2e|\xi|}\mathbf{e}, & \mathbf{e} \neq \mathbf{0} \\ \mathbf{r}_C^0 + \Delta\mathbf{r}, & \mathbf{e} = \mathbf{0} \end{cases}; \qquad \mathbf{b} = \begin{cases} \frac{1}{e\sqrt{2|\xi|}}(\mathbf{h} \times \mathbf{e}), & \mathbf{e} \neq \mathbf{0}, \\ \frac{1}{n}(\mathbf{v}_C^0 + \Delta\mathbf{v} + \boldsymbol{\omega}(t_0) \times \Delta\mathbf{r}), & \mathbf{e} = \mathbf{0} \end{cases} \tag{84}$$

$E(t)$ represents the deputy spacecraft eccentric anomaly; it is the solution to the Kepler equation:

$$E(t) - e\sin E(t) = n(t - t_P), \qquad t \in [t_0, +\infty) \tag{85}$$

where t_P denotes the time of periapsis passage of the deputy spacecraft and it is computed from the algorithm [34]:

$$\cos E(t_0) = \frac{1}{e}\left(1 - n\frac{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|}{\sqrt{2|\xi|}}\right) \tag{86}$$

$$\sin E(t_0) = n\frac{\Delta\mathbf{v}\cdot(\mathbf{r}_C^0 + \Delta\mathbf{r})}{2e|\xi|}\left[1 - \frac{\boldsymbol{\omega}(t_0)\cdot\mathbf{h}}{\mu}\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|\right] \tag{87}$$

$$t_P = t_0 - \frac{1}{n}[E(t_0) - e\sin E(t_0)] \tag{88}$$

Based on Eqs. (61), (62), (81), and (82), the relative law of motion and the relative velocity are modeled by

$$\begin{aligned}\mathbf{r} = {} & [\cos E(t) - e]\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\} + \sin E(t)\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right\} - \frac{p_C}{1 + e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|}\end{aligned} \tag{89}$$

$$\begin{aligned}\mathbf{v} = {} & \frac{-n\sin E(t)}{1 - e\cos E(t)}\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\} + \frac{n\cos E(t)}{1 - e\cos E(t)}\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C\right.\\ & \left. - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|} - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right\}\\ & + \frac{[1 + e_C\cos f_C(t)]^2[\cos E(t) - e]}{p_C^2}\\ & \times\left\{\frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C^2\mathbf{a} - \cos f_C^0(t)\tilde{\mathbf{h}}_C\mathbf{a}\right\}\\ & + \frac{[1 + e_C\cos f_C(t)]^2\sin E(t)}{p_C^2}\left\{\frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C^2\mathbf{b}\right.\\ & \left. - \cos f_C^0(t)\tilde{\mathbf{h}}_C\mathbf{b}\right\} - \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|}\end{aligned} \tag{90}$$

If the deputy trajectory is circular ($\mathbf{e} = 0$), Eqs. (84) are taken into account, together with

$$p = \|\mathbf{r}_C^0 + \Delta\mathbf{r}\|; \qquad E(t) = \frac{\|\mathbf{h}\|}{\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|^2}(t - t_0) \tag{91}$$

Equations (89) and (90) modify according to Eqs. (91).

If the reference trajectory is circular, the closed-form quaternionic Eqs. (89) and (90) change according to the following expressions [16,32]:

$$e_C = 0; \qquad f_C^0(t) = n_C(t - t_0) \tag{92}$$

It follows that in the situation when the chief spacecraft has an inertial circular trajectory, Eqs. (89) and (90) transform into

$$\begin{aligned}\mathbf{r} = {} & [\cos E(t) - e]\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\} + \sin E(t)\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C\right.\\ & \left. - \sin[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|} - \cos[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right\} - \mathbf{r}_C^0\end{aligned} \tag{93}$$

$$\begin{aligned}\mathbf{v} = {} & \frac{-n\sin E(t)}{1 - e\cos E(t)}\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\} + \frac{n\cos E(t)}{1 - e\cos E(t)}\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C\right.\\ & \left. - \sin[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|} - \cos[n_C(t - t_0)]\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right\}\\ & + \frac{\cos E(t) - e}{\|\mathbf{r}_C^0\|^2}\left\{\frac{1}{\|\mathbf{h}_C\|}\sin[n_C(t - t_0)]\tilde{\mathbf{h}}_C^2\mathbf{a} - \cos[n_C(t - t_0)]\tilde{\mathbf{h}}_C\mathbf{a}\right\}\\ & + \frac{\sin E(t)}{\|\mathbf{r}_C^0\|^2}\left\{\frac{1}{\|\mathbf{h}_C\|}\sin[n_C(t - t_0)]\tilde{\mathbf{h}}_C^2\mathbf{b} - \cos[n_C(t - t_0)]\tilde{\mathbf{h}}_C\mathbf{b}\right\}\end{aligned} \tag{94}$$

The solution offered by Eqs. (89) and (90) offer an alternative to the Lawden solution [2] to the linearized model of the relative orbital motion. Equations (93) and (94) offer an alternative to the solution to the linearized model of the relative orbital motion offered by Clohessy and Wiltshire [1].

In the end of this subsection, we will present the closed-form exact expressions for the relative law of motion and velocity with respect to the eccentric anomalies in the situation when both chief and deputy are satellites (the ellipse–ellipse situation). From the Kepler equations written for both chief and deputy inertial motions

$$E_C - e_C\sin E_C = n_C(t - t_P^C) \tag{95}$$

$$E - e\sin E = n(t - t_P) \tag{96}$$

one may derive the implicit equation that links these anomalies by eliminating the time t from Eqs. (95) and (96):

$$\frac{E_C - e_C\sin E_C}{n_C} + t_P^C = \frac{E - e\sin E}{n} + t_P \tag{97}$$

As the motion of the chief satellite is known, so is function E_C. The eccentric anomaly of the deputy satellite is then obtained by solving the implicit functional equation:

$$E - e\sin E = \frac{n}{n_C}(E_C - e_C\sin E_C) + n_C(t_P^C - t_P) \tag{98}$$

By taking into account the relations between the true anomaly and the eccentric anomaly of a Keplerian elliptic orbit

$$\begin{cases}\cos f = \frac{\cos E - e}{1 - e\cos E};\\ \sin f = \frac{\sqrt{1 - e^2}\sin E}{1 - e\cos E}\end{cases} \tag{99}$$

Equations (89) and (90) transform into

$$\begin{aligned}\mathbf{r} = {} & [\cos E - e]\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C\right.\\ & - \sqrt{1 - e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|}\\ & \left. - \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1 - e_C^2)\sin E_C\sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\}\\ & + \sin E\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C\right.\\ & - \sqrt{1 - e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|}\\ & \left. - \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1 - e_C^2)\sin E_C\sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right\}\\ & - \frac{p_C(1 - e_C\cos E_C)}{1 - e_C^2}\frac{\mathbf{r}_0}{\|\mathbf{r}_0\|}\end{aligned} \tag{100}$$

$$\mathbf{v} = \frac{-n\sin E}{1-e\cos E}\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sqrt{1-e_C^2}\frac{\sin(E_C^0+E_C)-e_C(\sin E_C+\sin E_C^0)}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|} - \frac{(\cos E_C-e_C)(\cos E_C^0-e_C)-(1-e_C^2)\sin E_C\sin E_C^0}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\}$$
$$+ \frac{n\cos E}{1-e\cos E}\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sqrt{1-e_C^2}\frac{\sin(E_C^0+E_C)-e_C(\sin E_C+\sin E_C^0)}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|} - \frac{(\cos E_C-e_C)(\cos E_C^0-e_C)-(1-e_C^2)\sin E_C\sin E_C^0}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right.$$
$$+ \frac{(1-e_C)^2(\cos E-e)}{(1-e_C\cos E_C)^2p_C^2} \times\left\{\sqrt{1-e_C^2}\frac{\sin(E_C^0+E_C)-e_C(\sin E_C+\sin E_C^0)}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C^2}{\|\mathbf{h}_C\|}\mathbf{a} - \frac{(\cos E_C-e_C)(\cos E_C^0-e_C)-(1-e_C^2)\sin E_C\sin E_C^0}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\tilde{\mathbf{h}}_C\mathbf{a}\right\}$$
$$+ \frac{(1-e_C)^2\sin E}{(1-e_C\cos E_C)^2p_C^2} \times\left\{\sqrt{1-e_C^2}\frac{\sin(E_C^0+E_C)-e_C(\sin E_C+\sin E_C^0)}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\frac{\tilde{\mathbf{h}}_C^2}{\|\mathbf{h}_C\|}\mathbf{b} - \frac{(\cos E_C-e_C)(\cos E_C^0-e_C)-(1-e_C^2)\sin E_C\sin E_C^0}{(1-e_C\cos E_C)(1-e_C\cos E_C^0)}\tilde{\mathbf{h}}_C\mathbf{b}\right\}$$
$$- \frac{e_C\|\mathbf{h}_C\|(1-e_C^2)\sin E_C}{(1-e_C\cos E_C)p_C}\frac{\mathbf{r}_0}{\|\mathbf{r}_0\|} \tag{101}$$

where $E_C^0 = E_C(t_0)$.

An expression similar to Eq. (100) was obtained by Ketema in [11].

B. Parabolic Inertial Deputy Trajectory ($\xi = 0$, $\mathbf{h} \neq 0$)

If the specific energy associated to the deputy inertial motion defined in Eq. (80) is zero, then this motion is parabolic (if its specific angular momentum is nonzero). The solution to the initial value problem (53) is [34]

$$\mathbf{r}_* = \frac{1}{2}(p-\mu\tau^2(t))\mathbf{e} + \tau(t)(\mathbf{h}\times\mathbf{e}) \tag{102}$$

and the expression of the vector quaternion $\dot{\mathbf{r}}_*$ is

$$\dot{\mathbf{r}}_* = \frac{2}{p+\mu\tau^2(t)}[-\mu\tau(t)\mathbf{e} + \mathbf{h}\times\mathbf{e}] \tag{103}$$

where p represents the deputy inertial orbit semilatus rectum:

$$p = \frac{\|\mathbf{h}\|^2}{\mu} \tag{104}$$

and the map $\tau(t)$ is the solution to the Kepler-like equation (also called Barker's equation [25]):

$$t - t_P = \frac{p}{2}\tau + \frac{\mu}{6}\tau^3 \tag{105}$$

Equation (105) has the solution:

$$\tau(t) = \frac{1}{\sqrt[3]{\mu}}\left[\sqrt[3]{3(t_P-t)+\sqrt{9(t_P-t)^2+\frac{p^3}{\mu}}} + \sqrt[3]{3(t_P-t)-\sqrt{9(t_P-t)^2+\frac{p^3}{\mu}}}\right] \tag{106}$$

where the moment of time of the periapsis passage of the deputy spacecraft t_P may be computed in two steps [34]:

$$\tau(t_0) = \frac{(\mathbf{r}_C^0+\Delta\mathbf{r})\cdot\Delta\mathbf{v}}{\mu^2}[\mu - \|\mathbf{r}_C^0+\Delta\mathbf{r}\|(\boldsymbol{\omega}(t_0)\cdot\mathbf{h})] \tag{107}$$

$$t_P = t_0 - \frac{1}{2}\left[p\tau(t_0) + \frac{\mu}{3}\tau^3(t_0)\right] \tag{108}$$

Based on Eqs. (61), (62), (102), and (103), the relative law of motion and the relative velocity are modeled by

$$\mathbf{r}(t) = \frac{1}{2}(p-\mu\tau^2(t))\left\{\frac{\mathbf{e}\cdot\mathbf{h}_C}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C\mathbf{e} - \frac{\cos f_C^0(t)}{\|\mathbf{h}_C\|^2}\tilde{\mathbf{h}}_C^2\mathbf{e}\right\} + \tau(t)\left\{\frac{(\mathbf{h},\mathbf{e},\mathbf{h}_C)}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C\tilde{\mathbf{h}}\mathbf{e} - \frac{\cos f_C^0(t)}{\|\mathbf{h}_C\|^2}\tilde{\mathbf{h}}_C^2\tilde{\mathbf{h}}\mathbf{e}\right\} - \frac{p_C}{1+e_C\cos f_C(t)}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{109}$$

$$\mathbf{v}(t) = \frac{-2\mu\tau(t)}{p+\mu\tau^2(t)}\left\{\frac{\mathbf{e}\cdot\mathbf{h}_C}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C\mathbf{e} - \frac{\cos f_C^0(t)}{\|\mathbf{h}_C\|^2}\tilde{\mathbf{h}}_C^2\mathbf{e}\right\} - \frac{2}{p+\mu\tau^2(t)}\left\{\frac{(\mathbf{h},\mathbf{e},\mathbf{h}_C)}{\|\mathbf{h}_C\|^2}\mathbf{h}_C + \frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C\tilde{\mathbf{h}}\mathbf{e} + \frac{\cos f_C^0(t)}{\|\mathbf{h}_C\|^2}\tilde{\mathbf{h}}_C^2\tilde{\mathbf{h}}\mathbf{e}\right\}$$
$$+ \frac{[1+e_C\cos f_C(t)]^2[p-\mu\tau^2(t)]}{2p_C^2}\left\{\frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C^2\mathbf{e} - \cos f_C^0(t)\tilde{\mathbf{h}}_C\mathbf{e}\right\} + \frac{[1+e_C\cos f_C(t)]^2\tau(t)}{p_C^2}\left\{\frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C^2\tilde{\mathbf{h}}\mathbf{e} - \cos f_C^0(t)\tilde{\mathbf{h}}_C\tilde{\mathbf{h}}\mathbf{e}\right\} - \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C}\frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|} \tag{110}$$

C. Hyperbolic Inertial Deputy Trajectory ($\xi > 0$, $\mathbf{h} \neq 0$)

If the specific energy associated to the deputy inertial motion defined in Eq. (80) is strictly positive, then the trajectory is hyperbolic (if the specific angular momentum is nonzero). The solution to the initial value problem (53) is [34]

$$\mathbf{r}_* = \mathbf{a}[e-\cosh E(t)] + \mathbf{b}\sinh E(t) \tag{111}$$

and the expression of the vector quaternion $\dot{\mathbf{r}}_*$ is

$$\dot{\mathbf{r}}_* = \frac{n}{e\cosh E(t)-1}[-\sinh E(t)\mathbf{a} + \cosh E(t)\mathbf{b}] \tag{112}$$

where $\mathbf{a}$ and $\mathbf{b}$ represent the vectorial semimajor and semiminor axis of the hyperbolic deputy trajectory, respectively; their expressions are

$$\mathbf{a} = \frac{\mu}{2e\xi}\mathbf{e}; \qquad \mathbf{b} = \frac{1}{e\sqrt{2\xi}}(\mathbf{h}\times\mathbf{e}) \tag{113}$$

while n has the expression

$$n = \frac{(2\xi)^{\frac{3}{2}}}{\mu} \tag{114}$$

$E(t)$ represents the eccentric anomaly of the deputy inertial trajectory; it is the solution to the Kepler-like equation:

$$E(t) - e \sinh E(t) = n(t - t_P), \qquad t \in [t_0, +\infty) \tag{115}$$

where t_P represents the moment of time corresponding to the periapsis passage of the deputy spacecraft; it may be computed by using the algorithm [34]:

$$E(t_0) = \sinh^{-1}\left\{\frac{n[\Delta\mathbf{v}\cdot(\mathbf{r}_C^0 + \Delta\mathbf{r})]}{2e\xi}\left[1 - \frac{\boldsymbol{\omega}(t_0)\cdot\mathbf{h}}{\mu}\|\mathbf{r}_C^0 + \Delta\mathbf{r}\|\right]\right\} \tag{116}$$

$$t_P = t_0 - \frac{1}{n}[E(t_0) - e \sinh E(t_0)] \tag{117}$$

where $\sinh^{-1}$ represents the inverse of the bijective function sinh:

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) \tag{118}$$

Based on Eqs. (61), (62), (116), and (117), the relative law of motion and the relative velocity are modeled by the vector quaternions:

$$\begin{aligned}\mathbf{r}(t) = {}& [e - \cosh E(t)]\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\} + \sinh E(t)\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right\} - \frac{p_C}{1 + e_C\cos f_C(t)}\frac{\mathbf{r}_0}{\|\mathbf{r}_0\|}\end{aligned} \tag{119}$$

$$\begin{aligned}\mathbf{v}(t) = {}& \frac{-n\sinh E(t)}{e\cosh E(t) - 1}\left\{\frac{\mathbf{h}_C\cdot\mathbf{a}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{a}}{\|\mathbf{h}_C\|}\right.\\ & \left. - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{a}}{\|\mathbf{h}_C\|^2}\right\} + \frac{n\cosh E(t)}{e\cosh E(t) - 1}\left\{\frac{\mathbf{h}_C\cdot\mathbf{b}}{\|\mathbf{h}_C\|^2}\mathbf{h}_C\right.\\ & \left. - \sin f_C^0(t)\frac{\tilde{\mathbf{h}}_C\mathbf{b}}{\|\mathbf{h}_C\|} - \cos f_C^0(t)\frac{\tilde{\mathbf{h}}_C^2\mathbf{b}}{\|\mathbf{h}_C\|^2}\right\}\\ & + \frac{[1 + e_C\cos f_C(t)]^2[e - \cosh E(t)]}{p_C^2}\\ & \times\left\{\frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C^2\mathbf{a} - \cos f_C^0(t)\tilde{\mathbf{h}}_C\mathbf{a}\right\}\\ & + \frac{[1 + e_C\cos f_C(t)]^2\sinh E(t)}{p_C^2}\left\{\frac{\sin f_C^0(t)}{\|\mathbf{h}_C\|}\tilde{\mathbf{h}}_C^2\mathbf{b}\right.\\ & \left. - \cos f_C^0(t)\tilde{\mathbf{h}}_C\mathbf{b}\right\} - \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C}\frac{\mathbf{r}_0}{\|\mathbf{r}_0\|}\end{aligned} \tag{120}$$

VI. Conclusions

The quaternionic approach used in this paper allows us to obtain closed-form exact expressions for the relative law of motion and the relative velocity. This instrument is only a catalyst, and it helps introduce a change of variable which transforms the relative orbital motion problem into the classic Kepler problem.

The shape of the chief inertial trajectory does not impose special problems, as it does in the linearized approaches. The deputy trajectory does not impose problems either, allowing us to derive exact equations of relative motion in any situation and for any initial conditions. The equations that describe the state of the deputy spacecraft in LVLH depend only on time and the initial conditions.

The long-term accuracy offered by this solution allows the study of the relative motion for indefinite time intervals, and with no restrictions on the magnitude of the relative distance. The solution may be used in the study of satellite constellations from the point of view of the relative motion.

The solution offered in this paper gives a parameterization of the manifold associated to the relative motion. Perturbation techniques may be now used in order to derive more accurate equations of motion when assuming small perturbations on the relative trajectory, due to Earth oblateness, solar wind, moon attraction, and atmospheric drag. Based on this solution, a study of the full-body relative motion might be a subject for future papers.

Appendix

We present here the scalar Cartesian expressions for the relative position and relative velocity as they are deduced from the quaternionic expressions presented in the paper. By denoting $\mathbf{r} = [x \quad y \quad z]^T$ the relative position vector, below we present the closed-form expressions for $x, y, z, \dot{x}, \dot{y}, \dot{z}$. We denote $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_z$ the unit vectors that define the axes of the LVLH frame; their expressions are

$$\mathbf{u}_x = \frac{\mathbf{r}_C^0}{\|\mathbf{r}_C^0\|}; \qquad \mathbf{u}_y = \frac{\tilde{\mathbf{h}}_C\mathbf{r}_C^0}{\|\mathbf{h}_C\|\|\mathbf{r}_C^0\|}; \qquad \mathbf{u}_z = \frac{\mathbf{h}_C}{\|\mathbf{h}_C\|} \tag{A1}$$

I. Situation $\xi < 0$, $\mathbf{h} \neq 0$: Elliptic Deputy Inertial Trajectory

$$\begin{aligned}x(t) = {}& [\cos E(t) - e]\{(\mathbf{u}_x\cdot\mathbf{a})\cos f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{a})\sin f_C^0(t)\}\\ & + \sin E(t)\{(\mathbf{u}_x\cdot\mathbf{b})\cos f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\sin f_C^0(t)\}\\ & - \frac{p_C}{1 + e_C\cos f_C(t)}\end{aligned} \tag{A2}$$

$$\begin{aligned}y(t) = {}& [\cos E(t) - e]\{-(\mathbf{u}_x\cdot\mathbf{a})\sin f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{a})\cos f_C^0(t)\}\\ & + \sin E(t)\{-(\mathbf{u}_x\cdot\mathbf{b})\sin f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\cos f_C^0(t)\}\end{aligned} \tag{A3}$$

$$z(t) = [\cos E(t) - e](\mathbf{u}_z\cdot\mathbf{a}) + \sin E(t)(\mathbf{u}_z\cdot\mathbf{b}) \tag{A4}$$

$$\begin{aligned}\dot{x}(t) = {}& \frac{n\sin E(t)}{1 - e\cos E(t)}\{(\mathbf{u}_x\cdot\mathbf{a})\cos f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{a})\sin f_C^0(t)\}\\ & + \frac{n\cos E(t)}{1 - e\cos E(t)}\{(\mathbf{u}_x\cdot\mathbf{b})\cos f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\sin f_C^0(t)\}\\ & - \frac{\mu[1 + e_C\cos f_C(t)]^2[\cos E(t) - e]}{\|\mathbf{h}_C\|}\{-(\mathbf{u}_x\cdot\mathbf{a})\sin f_C^0(t)\\ & + (\mathbf{u}_y\cdot\mathbf{a})\cos f_C^0(t)\} - \frac{\mu[1 + e_C\cos f_C(t)]^2\sin E(t)}{\|\mathbf{h}_C\|}\\ & \{-(\mathbf{u}_x\cdot\mathbf{b})\sin f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\cos f_C^0(t)\} - \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C}\end{aligned} \tag{A5}$$

$$\begin{aligned}\dot{y}(t) = {}& \frac{n\sin E(t)}{1 - e\cos E(t)}\{(\mathbf{u}_x\cdot\mathbf{a})\sin f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{a})\cos f_C^0(t)\}\\ & - \frac{n\cos E(t)}{1 - e\cos E(t)}\{-(\mathbf{u}_x\cdot\mathbf{b})\sin f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\cos f_C^0(t)\}\\ & - \frac{\|\mathbf{h}_C\|[1 + e_C\cos f_C(t)]^2[\cos E(t) - e]}{p_C}\{(\mathbf{u}_y\cdot\mathbf{a})\sin f_C^0(t)\\ & + (\mathbf{u}_x\cdot\mathbf{a})\cos f_C^0(t)\} - \frac{\|\mathbf{h}_C\|[1 + e_C\cos f_C(t)]^2\sin E(t)}{p_C}\\ & \times\{(\mathbf{u}_y\cdot\mathbf{b})\sin f_C^0(t) + (\mathbf{u}_x\cdot\mathbf{b})\cos f_C^0(t)\}\end{aligned} \tag{A6}$$

$$\dot{z}(t) = \frac{n}{[1 - e\cos E(t)]}[-\sin E(t)(\mathbf{u}_z \cdot \mathbf{a}) + \cos E(t)(\mathbf{u}_z \cdot \mathbf{b})] \tag{A7}$$

When the deputy trajectory is also an ellipse and one expresses the equations of the relative motion with respect to both eccentric anomalies, Eqs. (A2–A7) above transform into

$$\begin{aligned} x(t) &= [\cos E(t) - e] \\ &\times \left\{ \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{a}) \right. \\ &\left. - \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{a}) \right\} \\ &+ \sin E(t) \left\{ \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)} \right. \\ &\left. \times (\mathbf{u}_x \cdot \mathbf{b}) - \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{b}) \right\} \\ &- \frac{p_C(1 - e_C\cos E_C)}{1 - e_C^2} \end{aligned} \tag{A8}$$

$$\begin{aligned} y(t) &= -[\cos E(t) - e] \left\{ (\mathbf{u}_x \cdot \mathbf{a}) \sin f_C^0(t) \right. \\ &\left. + \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{a}) \right\} \\ &- \sin E(t) \left\{ \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{b}) \right. \\ &\left. + \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{b}) \right\} \end{aligned} \tag{A9}$$

$$z(t) = [\cos E(t) - e](\mathbf{u}_z \cdot \mathbf{a}) + \sin E(t)(\mathbf{u}_z \cdot \mathbf{b}) \tag{A10}$$

$$\begin{aligned} \dot{x}(t) &= \frac{n \sin E(t)}{1 - e\cos E(t)} \\ &\times \left\{ \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{a}) \right. \\ &\left. - \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{a}) \right\} \\ &+ \frac{n \cos E(t)}{1 - e\cos E(t)} \\ &\times \left\{ \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{b}) \right. \\ &\left. - \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{b}) \right\} \\ &- \frac{\mu[1 + e_C\cos f_C(t)]^2[\cos E(t) - e]}{\|\mathbf{h}_C\|} \\ &\times \left\{ \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{a}) \right. \\ &\left. + \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{a}) \right\} \\ &- \frac{\mu[1 + e_C\cos f_C(t)]^2 \sin E(t)}{\|\mathbf{h}_C\|} \\ &\times \left\{ \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{b}) \right. \\ &\left. + \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{b}) \right\} \\ &- \frac{e_C\|\mathbf{h}_C\|(1 - e_C^2)\sin E_C}{(1 - e_C\cos E_C)p_C} \end{aligned} \tag{A11}$$

$$\begin{aligned} \dot{y}(t) &= -\frac{n \sin E(t)}{1 - e\cos E(t)} \\ &\times \left\{ -\sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{a}) \right. \\ &\left. - \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{a}) \right\} \\ &- \frac{n \cos E(t)}{1 - e\cos E(t)} \\ &\times \left\{ \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{b}) \right. \\ &\left. - \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{b}) \right\} \\ &+ \frac{\|\mathbf{h}_C\|[1 + e_C\cos f_C(t)]^2[\cos E(t) - e]}{p_C} \\ &\times \left\{ \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{a}) \right. \\ &\left. + \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{a}) \right\} \\ &- \frac{\|\mathbf{h}_C\|[1 + e_C\cos f_C(t)]^2 \sin E(t)}{p_C} \\ &\times \left\{ \sqrt{1-e_C^2}\frac{\sin(E_C^0 + E_C) - e_C(\sin E_C + \sin E_C^0)}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_y \cdot \mathbf{b}) \right. \\ &\left. + \frac{(\cos E_C - e_C)(\cos E_C^0 - e_C) - (1-e_C^2)\sin E_C \sin E_C^0}{(1 - e_C\cos E_C)(1 - e_C\cos E_C^0)}(\mathbf{u}_x \cdot \mathbf{b}) \right\} \end{aligned} \tag{A12}$$

$$\dot{z}(t) = \frac{n}{[1 - e\cos E(t)]}[-\sin E(t)(\mathbf{u}_z \cdot \mathbf{a}) + \cos E(t)(\mathbf{u}_z \cdot \mathbf{b})] \tag{A13}$$

II. Situation $\xi = 0$, $\mathbf{h} \neq 0$: Parabolic Deputy Inertial Trajectory

$$\begin{aligned} x(t) &= \tfrac{1}{2}[p - \mu\tau^2(t)]\{(\mathbf{u}_x \cdot \mathbf{e})\cos f_C^0(t) + (\mathbf{u}_y \cdot \mathbf{e})\sin f_C^0(t)\} \\ &+ \tau(t)\{(\mathbf{h}, \mathbf{e}, \mathbf{u}_x)\cos f_C^0(t) + (\mathbf{h}, \mathbf{e}, \mathbf{u}_y)\sin f_C^0(t)\} \\ &- \frac{p_C}{1 + e_C\cos f_C(t)} \end{aligned} \tag{A14}$$

$$\begin{aligned} y(t) &= -\frac{1}{2}[p - \mu\tau^2(t)]\{(\mathbf{u}_x \cdot \mathbf{e})\sin f_C^0(t) - (\mathbf{u}_y \cdot \mathbf{e})\cos f_C^0(t)\} \\ &+ \tau(t)\{(\mathbf{h}, \mathbf{e}, \mathbf{u}_y)\cos f_C^0(t) - (\mathbf{h}, \mathbf{e}, \mathbf{u}_x)\sin f_C^0(t)\} \end{aligned} \tag{A15}$$

$$z(t) = \frac{1}{2}[p - \mu\tau^2(t)](\mathbf{u}_z \cdot \mathbf{e}) + \tau(t)(\mathbf{h}, \mathbf{e}, \mathbf{u}_z) \tag{A16}$$

$$\begin{aligned} \dot{x}(t) &= -\frac{2\mu\tau(t)}{p + \mu\tau^2(t)}\{(\mathbf{u}_x \cdot \mathbf{e})\cos f_C^0(t) + (\mathbf{u}_y \cdot \mathbf{e})\sin f_C^0(t)\} \\ &+ \frac{2}{p + \mu\tau^2(t)}\{(\mathbf{h}, \mathbf{e}, \mathbf{u}_x)\cos f_C^0(t) + (\mathbf{h}, \mathbf{e}, \mathbf{u}_y)\sin f_C^0(t)\} \\ &- \frac{\|\mathbf{h}_C\|[1 + e_C\cos f_C(t)]^2[p - \mu\tau^2(t)]}{2p_C}\{(\mathbf{u}_x \cdot \mathbf{e})\sin f_C^0(t) \\ &- (\mathbf{u}_y \cdot \mathbf{e})\cos f_C^0(t)\} - \frac{\|\mathbf{h}_C\|[1 + e_C\cos f_C(t)]^2\tau(t)}{p_C} \\ &\times \{(\mathbf{h}, \mathbf{e}, \mathbf{u}_x)\sin f_C^0(t) - (\mathbf{h}, \mathbf{e}, \mathbf{u}_y)\cos f_C^0(t)\} \\ &- \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C} \end{aligned} \tag{A17}$$

$$\dot{y}(t) = \frac{2\mu\tau(t)}{p + \mu\tau^2(t)}\{(\mathbf{u}_x\cdot\mathbf{e})\sin f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{e})\cos f_C^0(t)\} + \frac{2}{p+\mu\tau^2(t)}\{(\mathbf{h},\mathbf{e},\mathbf{u}_y)\cos f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{e})\sin f_C^0(t)\} + \frac{\|\mathbf{h}_C\|[1+e_C\cos f_C(t)]^2[p-\mu\tau^2(t)]}{2p_C}\{(\mathbf{u}_y\cdot\mathbf{e})\sin f_C^0(t) + (\mathbf{u}_x\cdot\mathbf{e})\cos f_C^0(t)\} + \frac{\|\mathbf{h}_C\|[1+e_C\cos f_C(t)]^2\tau(t)}{p_C} \times \{(\mathbf{h},\mathbf{e},\mathbf{u}_y)\sin f_C^0(t) + (\mathbf{h},\mathbf{e},\mathbf{u}_x)\cos f_C^0(t)\} \tag{A18}$$

$$\dot{z}(t) = \frac{2}{p+\mu\tau^2(t)}[-\mu\tau(t)(\mathbf{u}_x\cdot\mathbf{e}) + (\mathbf{h},\mathbf{e},\mathbf{u}_x)] \tag{A19}$$

III. Situation $\xi > 0$, $\mathbf{h} \neq 0$: Hyperbolic Deputy Inertial Trajectory

$$x(t) = [e - \cosh E(t)]\{(\mathbf{u}_x\cdot\mathbf{a})\cos f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{a})\sin f_C^0(t)\} + \sinh E(t)\{(\mathbf{u}_x\cdot\mathbf{b})\cos f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\sin f_C^0(t)\} - \frac{p_C}{1+e_C\cos f_C(t)} \tag{A20}$$

$$y(t) = -[e - \cosh E(t)]\{(\mathbf{u}_x\cdot\mathbf{a})\sin f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{a})\cos f_C^0(t)\} - \sinh E(t)\{(\mathbf{u}_x\cdot\mathbf{b})\sin f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{b})\cos f_C^0(t)\} \tag{A21}$$

$$z(t) = [e - \cosh E(t)](\mathbf{u}_z\cdot\mathbf{a}) + \sinh E(t)(\mathbf{u}_z\cdot\mathbf{b}) \tag{A22}$$

$$\dot{x}(t) = -\frac{n\sinh E(t)}{e\cosh E(t) - 1}\{(\mathbf{u}_x\cdot\mathbf{a})\cos f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{a})\sin f_C^0(t)\} + \frac{n\cosh E(t)}{e\cosh E(t)-1}\{(\mathbf{u}_x\cdot\mathbf{b})\cos f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\sin f_C^0(t)\} - \frac{\|\mathbf{h}_C\|[1+e_C\cos f_C(t)]^2[e-\cosh E(t)]}{p_C}\{(\mathbf{u}_x\cdot\mathbf{a})\sin f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{a})\cos f_C^0(t)\} - \frac{\|\mathbf{h}_C\|[1+e_C\cos f_C(t)]^2\sinh E(t)}{p_C} \times \{(\mathbf{u}_x\cdot\mathbf{b})\sin f_C^0(t) - (\mathbf{u}_y\cdot\mathbf{b})\cos f_C^0(t)\} - \frac{e_C\|\mathbf{h}_C\|\sin f_C(t)}{p_C} \tag{A23}$$

$$\dot{y}(t) = -\frac{n\sinh E(t)}{e\cosh E(t)-1}\{(\mathbf{u}_x\cdot\mathbf{a})\sin f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{a})\cos f_C^0(t)\} - \frac{n\cosh E(t)}{e\cos E(t)-1}\{(\mathbf{u}_x\cdot\mathbf{b})\sin f_C^0(t) + (\mathbf{u}_y\cdot\mathbf{b})\cos f_C^0(t)\} - \frac{\|\mathbf{h}_C\|[1+e_C\cos f_C(t)]^2[e-\cosh E(t)]}{p_C}\{(\mathbf{u}_y\cdot\mathbf{a})\sin f_C^0(t) + (\mathbf{u}_x\cdot\mathbf{a})\cos f_C^0(t)\} - \frac{\|\mathbf{h}_C\|[1+e_C\cos f_C(t)]^2\sinh E(t)}{p_C} \times \{(\mathbf{u}_y\cdot\mathbf{b})\sin f_C^0(t) + (\mathbf{u}_x\cdot\mathbf{b})\cos f_C^0(t)\} \tag{A24}$$

$$\dot{z}(t) = \frac{n}{[e\cosh E(t) - 1]}[-\sinh E(t)(\mathbf{u}_x\cdot\mathbf{a}) + \cosh E(t)(\mathbf{u}_z\cdot\mathbf{b})] \tag{A25}$$

Note that in all cases, the out-of-plane component of the relative motion is completely decoupled from the in-plane component, in the sense that it does not depend on the chief true anomaly.

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